5.3 a.

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Ex. 5.3., 5.1, 5.4
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Show that:

$$\sum_{i=0}^{n} L_i(t) = 1 \quad \forall t \in \mathbb{R}.$$

Proof: Since $p(x) = \sum_{i=0}^{n} y_i L_i(x)$ set $y_i = 1$, i.e. define the polynomial Lagrange interpolant p through the data points $(t_i, 1)_{i=0}^{n}$

$$\Rightarrow p(x) = \sum_{i=0}^{n} 1 \cdot L_i(x)$$

$\Rightarrow$ Now define p s.t.

$$p(t_i) = 1 \quad \text{for } i = 0, \dots, n$$

$\Rightarrow$ $p \equiv 1$ is the interpolating polynomial for $(t_i, 1)_{i=0}^{n}$

$\Rightarrow$ (theorem 5.2.14) $p \equiv 1$ is unique solution for Lagrange polynomial interpolation problem

$$\Rightarrow 1 = p(x) = \sum_{i=0}^{n} L_i(x) \quad \blacksquare$$

5.3.c

Show that:

$$\sum_{i=0}^{n} y_i L_i(x) = \omega(x) \sum_{i=0}^{n} \frac{y_i}{(x-t_i)} \omega'(t_i)$$

$$\text{with } \omega(t) := \prod_{j=0}^{n} (t-t_j).$$

Proof:

$$\omega'(x) = \frac{d}{dx} \prod_{j=0}^{n} (x-t_j) = \sum_{i=0}^{n} \prod_{\substack{j=0 \\ j\neq i}}^{n} (x-t_j)$$

$$\omega'(t_1) = \sum_{i=0}^{n} \prod_{\substack{j=0 \\ i\neq j}}^{n} (t_1-t_j).$$

$$\begin{aligned}
= \; & \overbrace{(t_1-t_1)\cdot(t_1-t_2)\cdots(t_1-t_n)}^{=0} & [i=0] \\
+ \; & \boxed{(t_1-t_0)(t_1-t_2)\cdots(t_1-t_n)} & [i=1] \\
+ \; & \underbrace{(t_1-t_0)(t_1-t_1)(t_1-t_3)\cdots(t_1-t_n)}_{=0} & [i=2] \\
& & \vdots
\end{aligned}$$

$$\Rightarrow \omega'(t_i) = \prod_{\substack{i=0 \\ j\neq i}}^{n} (t_i-t_j) \quad \text{since } (x-t_j) = 0 \text{ for } x = t_j \,!$$

$$\Rightarrow \quad p(x) = \omega(x) \sum_{i=0}^{n} \frac{y_i}{(x-t_i)\,\omega'(t_i)}$$

$$= \sum_{i=0}^{n} \left[\frac{y_i}{(x-t_i) \prod\limits_{\substack{j=0 \\ j\neq i}}^{n} (t_i - t_j)} \cdot \prod_{j=0}^{n} (x - t_j) \right]$$

(cancel $(x-t_i)$ from $\prod\limits_{j=0}^{n}(x-t_j)$, since $i=0,\dots,n$ and $j=0,\dots,n$, meaning that $(x-t_i)$ is a factor in $\prod\limits_{j=0}^{n}(x-t_j)$)

$$= \sum_{i=0}^{n} \frac{y_i}{\prod\limits_{\substack{j=0 \\ j\neq i}} (t_i - t_j)} \cdot \prod_{\substack{j=0 \\ \boxed{j\neq i}}}^{n} (x - t_j)$$

$$= \sum_{i=0}^{n} y_i \cdot \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{(x-t_j)}{(t_i - t_j)}$$

$$= \sum_{i=0}^{n} y_i \cdot L_i(x) \quad \blacksquare$$

5.1.a.

Vector c contains the coefficients of the polynomials in the monomial basis, with the leading coeff. in c(0).

Ex.: $x^3 - 2x + 3$

$$\Rightarrow c = [1, 0, -2, 3]^T$$

Horner Scheme:

$$p(t) = t(\dots t(t(\alpha_n t + \alpha_{n-1}) + \alpha_{n-2} + \dots + \alpha_1) + \alpha_0$$

$$= \alpha_n t^n + \alpha_{n-1} t^{n-1} + \dots + \alpha_1 x + \alpha_0$$

$$p'(t) = n \cdot \alpha_n t^{n-1} + (n-1) \cdot \alpha_{n-1} t^{n-2} + \dots + \alpha_1$$

(Horner Scheme)

$$= t(\dots t(t(n \cdot \alpha_n \cdot t + (n-1) \cdot \alpha_{n-1}) + (n-2)\alpha_{n-2}$$

$$+ \dots + 2\alpha_2) + \alpha_1$$

5.1.e. Interpolation points: (t_i, y_i), $i = 0, \dots, n$.

$$p_{i,i}(t) \equiv y_i \qquad i = 0, \dots, n$$

$$\Rightarrow \quad p'_{i,i}(t) \equiv 0 \qquad i = 0, \dots, n$$

$$p_{0,n}(t) = \frac{(t - t_0)\, p_{1,n}(t) - (t - t_n)\, p_{0,n-1}(t)}{t_n - t_0}$$

$$\Rightarrow \quad p'_{0,n}(t) = \frac{p_{1,n}(t) + (t - t_0)\, p'_{1,n}(t) - \left[p_{0,n-1}(t) + (t - t_n) \cdot p'_{0,n-1}(t) \right]}{t_n - t_0}$$

<u>Example</u>: Evaluate $p_{0,3}(x)$ at $x \in \mathbb{R}$

(Aitken-Neville Algorithm)

$$\begin{array}{ccccccc}
p_{0,0}(x) = y_0 & \searrow & & & & & \\
 & & p_{0,1}(x) & \searrow & & & \\
p_{1,1}(x) = y_1 & \nearrow\searrow & & & p_{0,2}(x) & \searrow & \text{RESULT} \\
 & & p_{1,2}(x) & \nearrow\searrow & & & \boxed{p_{0,3}(x)} \\
p_{2,2}(x) = y_2 & \nearrow\searrow & & & p_{1,3}(x) & \nearrow & \\
 & & p_{2,3}(x) & \nearrow & & & \\
p_{3,3}(x) = y_3 & \nearrow & & & & &
\end{array}$$

Solution Code 5.0.4 explained

Example: again, eval. $p_{0,3}(x)$ at $x \in \mathbb{R}$

$p := y \Rightarrow p := \boxed{y_0 \mid y_1 \mid y_2 \mid y_3}$

1. Step:

$i_m = 1$ (upper bound)

$\Rightarrow i_0 = 0$ (lower bound)

$\Rightarrow$ set $p[0] := p_{0,1}(x)$.

$\Rightarrow p := \boxed{p_{0,1}(x) \mid y_1 \mid y_2 \mid y_3}$

$$\left(\begin{array}{ccc} p_{0,0}(x) = y_0 & & \\ & & p_{0,1}(x) \\ p_{1,1}(x) = y_1 & & \end{array} \right.$$

After the first step we don't need $p_{0,0}$ any more, so it gets overwritten with $p_{0,1}(x)$!

2. Step:

$i_m = 2$

$\Rightarrow i_0 = 1, 0$ (in this order!)

$\Rightarrow$ set $p[1] := p_{1,2}(x)$
$p[0] := p_{0,2}(x)$

$\Rightarrow p := \boxed{p_{0,2}(x) \mid p_{1,2}(x) \mid y_2 \mid y_3}$

3. Step:

$i_m = 3$

$\Rightarrow i_0 = 2, 1, 0$

$\Rightarrow$ set: $p[2] := p_{2,3}(x)$, $p[1] := p_{1,3}(x)$
$p[0] = \underline{\underline{p_{0,3}(x)}}$

$\Rightarrow p := \boxed{p_{0,3}(x) \mid p_{1,3}(x) \mid \ldots}$

$\Rightarrow p[0]$ contains our result!

5.4. Hermite polynomial seeks

a polynomial $p \in \mathcal{P}_{2n+1}$ satisfying

$$p(t_i) = y_i \quad \text{and} \quad p'(t_i) = c_i,$$

$$i = 0, \dots, n$$

$$\text{and} \quad p'(t) = \frac{d}{dt} p(t)$$

Hermite interpolant in monomial basis:

$$\underbrace{y_j = p(t_j) = \sum_{i=0}^{2n+1} a_i t_j^i, \quad j = 0, \dots, n}_{\text{first } (n+1) \text{ equations.}}$$

$$c_j = p'(t_j) = \frac{d}{dt_j} \sum_{i=0}^{2n+1} a_i t_j^i$$

$$\underbrace{= \sum_{\boxed{i=1}}^{2n+1} a_i \cdot i \cdot t_j^{i-1}, \quad j = 0, \dots, n.}_{\text{second } (n+1) \text{ equations.}}$$

We seek a system $Va = b$ with

$$a = [a_0, \dots, a_{2n+1}]^T$$

Set $b = [y_0, \dots, y_n, c_0, \dots, c_n]^T$

and
$$V = \underbrace{\begin{bmatrix} 1 & t_0 & t_0^2 & \cdots & t_0^{2n+1} \\ \vdots & & & & \\ 1 & t_n & t_n^2 & \cdots & t_n^{2n+1} \\ 0 & 1 & 2t_0 & \cdots & (2n+1)\, t_0^{2n} \\ \vdots & & & & \\ 0 & 1 & 2t_n & \cdots & (2n+1)\, t_n^{2n} \end{bmatrix}}_{\text{Vandermonde matrix}}$$