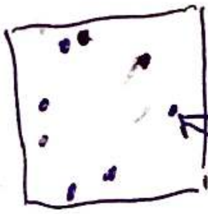
3.7

$p_i \in \mathbb{R}^2, \quad i = 1, \dots, n, \quad n = 8$

$x_i \in \mathbb{R}^2, \quad i = 1, \dots, 16$

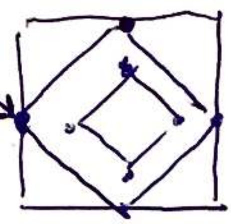
NumCSE 16
Ex. 3.7, 3.9
9.11.16
G. Accaputo

Input:  (Photograph)

p_i

$x_{i,prio}$

Priority sign

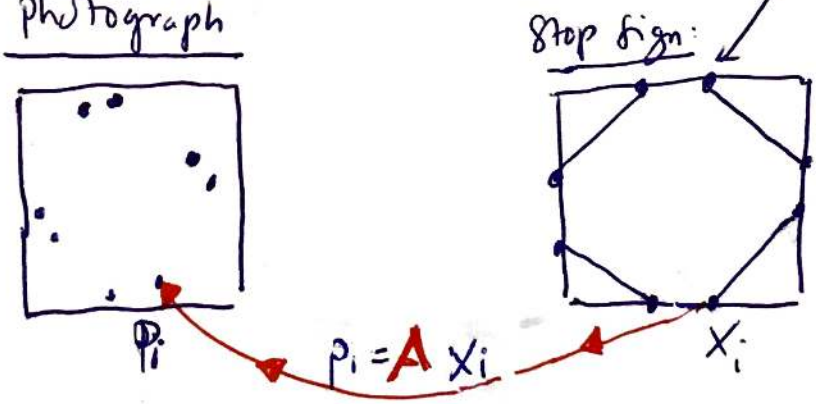


Output: It's a stop (or priority) sign.

Goal: Photograph is a result of a linear transformation of the original points x_i, i.e. there exists $A \in \mathbb{R}^{2 \times 2}$, s.t.

$$p^i = A x^i$$

$x_{i,stop}$



Use Least Squares Method to find the matrix A and to classify the given points

3.7.a.

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad x^i = \begin{pmatrix} x_1^i \\ x_2^i \end{pmatrix}$$

$$p^i = \begin{pmatrix} p_1^i \\ p_2^i \end{pmatrix}$$

$$p^i = A x_i^i \iff p_i^i = \begin{pmatrix} a_{11} \cdot x_1^i + a_{12} \cdot x_2^i \\ a_{21} \cdot x_1^i + a_{22} \cdot x_2^i \end{pmatrix}$$

Define overdetermined linear system

$$w = Bv$$

whose Least Squares solution will allow to define A.

$\Rightarrow$ all entries of A are unknown, thus:

$$v = \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix} = \mathrm{vec}(A^T)$$

$$\Rightarrow \quad p^i = \begin{bmatrix} x_1^i & x_2^i & 0 & 0 \\ 0 & 0 & x_1^i & x_2^i \end{bmatrix} \cdot \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix}$$

$$\Rightarrow \quad \underbrace{\begin{bmatrix} p_1^1 \\ p_2^1 \\ p_1^2 \\ p_2^2 \\ \vdots \\ p_1^n \\ p_2^n \end{bmatrix}}_{w :=} = \underbrace{\begin{bmatrix} x_1^1 & x_2^1 & 0 & 0 \\ 0 & 0 & x_1^1 & x_2^1 \\ x_1^2 & x_2^2 & 0 & 0 \\ 0 & 0 & x_1^2 & x_2^2 \\ & \vdots & & \vdots \\ x_1^n & x_2^n & 0 & 0 \\ 0 & 0 & x_1^n & x_2^n \end{bmatrix}}_{B_{\text{''}}} \cdot \underbrace{\begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{bmatrix}}_{v_{\text{''}}}$$

$B \in \mathbb{R}^{2n \times 4}$

$v \in \mathbb{R}^4$

$w \in \mathbb{R}^{2n}$

3.7.b

$\text{rank}(B) \leq \min(4, 2n) \Leftrightarrow \text{rank}(B) \leq 4$

$\text{rank}(B) = 4 \Leftrightarrow$ B has full rank

$\Leftrightarrow$ All 4 columns of B are linearly independent.

$\Leftrightarrow x_2^i \neq c \cdot x_1^i$, i.e., there exists no $c \in \mathbb{R}$ s.t. the colums of B are linearly dependent.

3.9. a

Show: A^TA admits a Cholesky decomposition

Proof: A^TA has to be symmetric positive-definite for it to admit a Cholesky decomposition. (see Lemma 2.8.14)

1.) Show: A^TA is symmetric.

Proof: $(A^TA)^T = A^T(A^T)^T = A^TA$

$\Rightarrow (A^TA)^T = A^TA$

$\Rightarrow$ A^TA is symmetric $\square$

2.) Show: A^TA is positive-definite. (p.d.)

Proof: From Def. 1.1.8 we know that A^TA is p.d. iff ← "if and only if"

$$\forall v \in \mathbb{R}^n, v \neq 0 : v^TA^TAv > 0$$

$$\Rightarrow v^TA^TAv = \underbrace{(Av)^T}_{\in \mathbb{R}^n} \underbrace{Av}_{\in \mathbb{R}^n}$$

$$= (Av) \cdot (Av)$$

dot product

$$= \|Av\|_2^2$$

$\Rightarrow$ Since $v \neq 0$ and $\|\cdot\|_2 \geq 0$ it follows that

$$\|Av\|_2^2 > 0 \Leftrightarrow v^TA^TAv > 0$$

From 1.) and 2.) it follows that A^TA is s.p.d.

$\Rightarrow$ (Lemma 2.8.14) A^TA admits a Cholesky decomp. $\blacksquare$

3.9. b.

CholeskyQR(...)

- $L = PP^T = A^TA$ $\Rightarrow$ AtA.LLt() calculats the Cholesky decomp. (CD) of A^TA

P is a lower triangular matrix!

- ! $R = P^T$ $\Rightarrow$ L.matrixL() retrieves the factor P of the above CD.
- $Q = (R^{-T}A^T)^T$ $\Rightarrow$ A.solve(b) calculats $A^{-1}b$

<u>Show:</u> Q, R calculated by CholeskyQR(...) are the same as Q, R calculated by a QR-decomposition

<u>Proof:</u> Theorem 3.3.9: Q has to be orthogonal, R upper triangular and

$$A = Q \cdot R.$$

1.) Show: Q is orthogonal, i.e. $Q^TQ = I$

Proof: $Q = (R^{-T}A^T)^T = AR^{-1}$

$$\Rightarrow Q^TQ = R^{-T}A^TAR^{-1}$$

$\Rightarrow$ (use $R^T = P$, $R^TR = PP^T = A^TA$)

$$Q^TQ = R^{-T}R^TRR^{-1}$$

$$\Rightarrow Q^TQ = I$$

$\Rightarrow$ Q is orthogonal

2) Show: R is upper triangular

Proof: $R = P^T$ and $L = PP^T$ = CD of A^TA

$\Rightarrow$ P is lower triangular matrix

$\Rightarrow$ P^T is upper triangular matrix $\Rightarrow$ $R = P^T$ is upper triangul.

3.) Show: $A = QR$

Proof: $QR = (R^{-T}A^T)^T R = AR^{-1}R = \underline{A}$

$\Rightarrow$ From 1.), 2.) and 3.) it follows that

CholeskyQR(...) calculates a QR-decomposition of A using a Cholesky decomposition ■