2.8 d.

$$A = E \cdot D \Rightarrow A^{-1} = (E \cdot D)^{-1} = D^{-1} E^{-1}$$

NumCSE 16.
26.10.16
Ex 2.8, 2.9, 2.12, 2.13
G. Accaputo.

with $D^{-1} = \begin{pmatrix} 1/a_1 & & 0 \\ & 1/a_2 & \\ 0 & & \ddots & \\ & & & 1/a_n \end{pmatrix}$

Calculate $E^{-1} \Rightarrow E E^{-1} = I$

$$\Rightarrow \left(\begin{array}{cccccc|ccccc} 1 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots \\ 1 & 1 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots \\ & & \vdots & & & & & & \vdots & & \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{cccccc|ccccc} 1 & 0 & 0 & 0 & 0 & & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & & -1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & \cdots & -1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 1 & 1 & 0 & \cdots & -1 & 0 & 0 & 1 & \cdots \\ & & \vdots & & & & & & \vdots & & \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{cccccc|ccccc} 1 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & 0 & \cdots & -1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 0 & & 0 & -1 & 0 & 1 & \cdots \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{cccccc|cccccc} 1 & 0 & 0 & 0 & 0 & & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots & -1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & & 0 & -1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & & 0 & 0 & -1 & 1 & 0 & \cdots \\ & & \vdots & & & & & & \vdots & & & \end{array}\right)$$

and so on...

$$\Rightarrow A^{-1} = D^{-1} \cdot E^{-1}$$

$$= \begin{bmatrix} 1/a_1 & 0 & 0 & 0 & & \\ -1/a_2 & \frac{1}{a_2} & 0 & 0 & \dots & \\ 0 & -1/a_3 & \frac{1}{a_3} & 0 & & \\ 0 & 0 & -1/a_4 & \frac{1}{a_4} & \dots & \\ & & \vdots & & & \\ & & & 0 & -\frac{1}{a_n} & \frac{1}{a_n} \end{bmatrix}$$

X can now be calculated!

$$\Rightarrow X = D^{-1} \cdot E^{-1} \cdot b$$

$$\Rightarrow X_j = \begin{cases} b_1/a_1 \,, & j = 1 \\ \dfrac{b_j - b_{j-1}}{a_j} \,, & j = 2, \dots, n \end{cases}$$

2.9. CRS-format:

$$A = \begin{pmatrix} 10 & 0 & -2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

row-ptr: | 1 | 4 | 6 | 8 | 9 |

$9 = nnz+1$

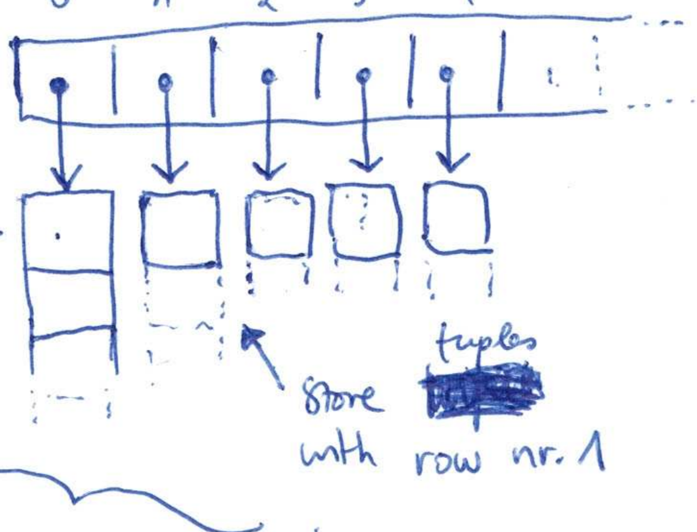
val: | 10 | -2 | 3 | 1 | 2 | 2 | 3 | 1 |

col-ind: | 1 | 3 | 4 | 3 | 4 | 2 | 4 | 4 |

2.9.d.

Use a "vector of vectors" to store the ~~[redacted]~~ (col, val) tuples:

row nr. → 0 1 2 3 4 ...



store ~~[redacted]~~ tuples with row nr. 0

store ~~[redacted]~~ tuples with row nr. 1

Vectors containing the ~~[redacted]~~ (col, val) tuples have to be sorted by the column nr. and cannot contain double entries:

$(1,1), (1,2), (1,3), \ldots, (m,n)$

(math notation, index thus starts at 1 here)

⇒ 2 possibilities:

1. Sorted insertion, i.e. insert ~~[redacted]~~ (col, val) already at the correct position
2. Insert all tuples, then sort it and sum up all the duplicated column values.

2.12.a.

Show: L is a linear operator

where $L: \mathbb{R}^{n,m} \to \mathbb{R}^{n,m}$

$$X_{ij} \mapsto \begin{cases} \sum_{k,l=1}^{3} S_{k,l} \, X_{i+k-2,\, j+l-2} & \text{if well-defined} \\ X_{ij} & \text{else.} \end{cases}$$

1i) show that $L(X+Y) = L(X) + L(Y)$.

$\Rightarrow L(X+Y)$ is defined as

$$x_{ij} + y_{ij} \mapsto \begin{cases} \sum_{k,l=1}^{3} S_{k,l} \left(x_{i+k-2,\, j+l-2} + y_{i+k-2,\, j+l-2} \right) \\ x_{ij} + y_{ij} \end{cases}$$

$$= \sum_{k,l=1}^{3} S_{k,l} \, x_{i+k-2,\, j+l-2} + S_{k,l} \, y_{i+k-2,\, j+l-2}$$

$$= \sum_{k,l=1}^{3} s_{k,l} \, x_{i+k-2,\, j+l-2} + \sum_{k,l}^{3} S_{k,l} \, y_{i+k-2,\, j+l-2}$$

$$\Rightarrow \quad L(X+Y), \quad x_{ij}+y_{ij} \mapsto \left\{ \begin{matrix} \underbrace{\begin{matrix} \sum_{k,l=1}^{3} s_{k,l}\, x_{i+k-2,j+l-2} \\ x_{ij} \end{matrix}}_{L(X)} & + & \underbrace{\begin{matrix} \sum_{k,l=1}^{3} s_{k,l}\, y_{i+k-2,j+l-2} \\ y_{ij} \end{matrix}}_{L(Y)} \end{matrix} \right.$$

$$\Rightarrow \quad L(X+Y) = L(X) + L(Y) \qquad \blacksquare$$

2.12.b.

Let $X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$

Find $A \in \mathbb{R}^{3\times 3}$ s.t.

$$A \cdot vec(X) = vec(L(X))$$

If we write out the sum in the definition of $L(X)$ we get:

$k=1$; $l=1$ $\quad s_{11} \cdot x_{i-1,j-1} = 0$

$l=2$ $\quad s_{12} \cdot x_{i-1,j}$

$l=3$ $\quad s_{13} \cdot x_{i-1,j+1} = 0$

$k=2$: $l=1$ $\quad s_{21} \cdot x_{i,j-1}$

$l=2$ $\quad s_{22} \cdot x_{i,j}$ with $S = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

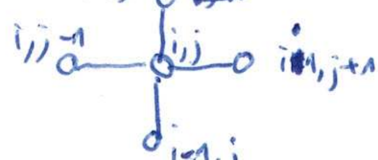
$l=3$ $\quad s_{23} \cdot x_{i,j+1}$

$k=3$: $l=1$ $\quad s_{31} \cdot x_{i+1,j-1} = 0$

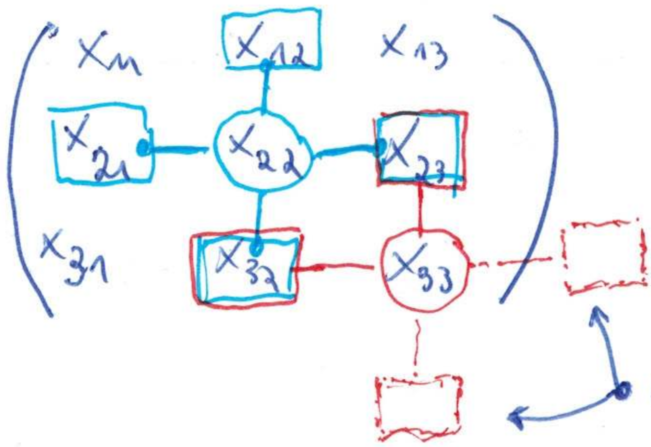
$l=2$ $\quad s_{32} \cdot x_{i+1,j}$

$l=3$ $\quad s_{33} \cdot x_{i+1,j+1} = 0$

$$\Rightarrow x_{ij} \longmapsto \begin{cases} \underbrace{s_{12} \cdot x_{i-1,j} + s_{21} \cdot x_{i,j-1} + \cancel{4} \cdot s_{22} \cdot x_{i,j} + s_{23} \cdot x_{i,j+1} + s_{32} \cdot x_{i+1,j}}_{\text{This is a 5-point stencil}} & \text{if well def.} \\ x_{ij}, & \text{otherwise.} \end{cases}$$



By applying this 5-point stencil on every entry of $\underline{X}$, we can see that L is only well-defined for the entry x_{22}:



not well-defined!

Thus, for $L(X)$ we get:

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & \boxed{\blacktriangle} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$

$x_{21} + x_{12} - 4x_{22} + x_{32} + x_{23}$

We have:

$$A \cdot \mathrm{vec}(X) = \mathrm{vec}(L(X))$$

$$\Rightarrow \quad A \cdot \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \\ x_{12} \\ x_{22} \\ x_{32} \\ x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \\ x_{12} \\ x_{21} + x_{12} - 4x_{22} + x_{32} + x_{23} \\ x_{32} \\ x_{13} \\ x_{23} \\ x_{33} \end{bmatrix}$$

$$\Rightarrow \quad A = \begin{array}{c} \begin{matrix} x_{11} & x_{21} & x_{31} & x_{12} & x_{22} & x_{32} & x_{13} & x_{23} & x_{33} \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

2.13.e

$$C_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$$

Example:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

A in COO format, sorted by column nr.:

$$\Rightarrow A = \{ (0,0,1), (1,0,1), (2,0,1), (1,1,1) \}$$

B in COO format, sorted by row nr.:

$$\Rightarrow B = \{ (0,1,1), (1,0,1), (1,2,1), (2,1,1) \}.$$

Obtain all possible triplets $a_{ik} \cdot b_{kj}$ by looping between all triplets of A and B which have k in common.

In our example above, the triplets $(0,0,1)$, $(1,0,1)$, and $(2,0,1)$ are multiplied with the triplet $(0,1,1)$ of B, resulting in the entries of C_{01}, C_{11}, and C_{21}.